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# An algebraic description of regular epimorphisms in topology

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**Abstract**

Recent work of Janelidze and Sobral on descent theory of finite topological spaces motivated our interest in ultrafilter descriptions of various classes of continuous maps. In earlier papers we presented such characterizations for triquotient maps and local homeomorphisms, here we do it for regular epimorphisms. To do so, we give an alternative description of the "obvious" reflection of pseudotopological spaces into topological spaces. Topological spaces, when presented as ultrafilter convergence structures, are examples of $(\mathsf{T};\mathbf{V})$-algebras introduced by Clementino and Tholen in "Metric, Topology and Multicategory—a Common Approach". In this paper, we work in this general setting and hence obtain at once characterizations of regular epimorphisms between topological spaces, approach spaces and (generalized) metric spaces, as well as the characterization for preordered sets which motivated our work.


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## 0. Introduction

In [9,10] the authors prove characterizations of various kinds of topological descent maps between finite topological spaces, "which become very simple and natural as soon as they are expressed in the language of finite preorders" [10]. Although these "finite results" are very helpful to understand and motivate the theory of topological descent and a great

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source for (counter-) examples, it is of interest to know their infinite extensions. Obviously, instead of considering the preorder relation one must study the convergence relation between ultrafilters and points, between ultrafilters of ultrafilters and ultrafilters and so on; hence one has to deal with a much more complicated situation. In our recent work we succeeded in the case of triquotient maps [2] and local homeomorphisms [4]. It is the purpose of the present paper to obtain the ultrafilter version of the following characterization of regular epimorphisms between preordered sets.

**Theorem.** *An order-preserving map $f : X \to Y$ is a regular epimorphism in* **Ord** *if and only if the order relation on Y can be obtained from "zigzags" in X; that is, for each $y_1 \to y_0$ in Y there is a "zigzag"*

$$\begin{tikzcd}
x_n \arrow[d] \\
x_{n-1} \arrow[r, phantom, "\sim_f"] & x'_{n-1} \arrow[d] \\
& x_{n-2} \arrow[r, phantom, "\sim_f"] & x'_{n-2} \arrow[d] \\
& & \cdots \arrow[r, dotted, no head] & \cdots \arrow[d] \\
& & & x_1 \arrow[r, phantom, "\sim_f"] & x'_1 \arrow[d] \\
& & & & x_0
\end{tikzcd}$$

*in X of length n, for some $n \in \mathbb{N}$, with $f(x_n) = y_1$ and $f(x_0) = y_0$, where $\sim_f$ denotes the kernel relation of f.*

In order to obtain a characterization of topological quotient maps in terms of ultrafilters we need a description of topological spaces in terms of their convergence structure. This is most elegantly expressed in [1] where topological spaces are presented as sets *X* equipped with a relation $\mathfrak{x} \to x$ between ultrafilters and points, subject to the *reflexivity* and the *transitivity* condition

$$e_X(x) = \dot{x} \to x \quad (\mathfrak{X} \to \mathfrak{x} \ \&\ \mathfrak{x} \to x) \Rightarrow m_X(\mathfrak{X}) \to x$$

for all $x \in X$, $\mathfrak{x} \in UX$ and $\mathfrak{X} \in UUX$. As Barr observed, these two conditions are exactly the laws of a lax Eilenberg–Moore algebra for the natural extension of the ultrafilter monad $\mathsf{U} = (U, e, m)$ to a lax monad on **Rel**.

A preorder *a* on a set *X* may also be viewed as an internal *monoid* in **Rel**: it is an endorelation *a* of *X* such that

$$\Delta_X \leqslant a, \quad a \cdot a \leqslant a.$$

In order to transport this idea to topological spaces, we introduce the *co-Kleisli composition* $a * b := a \cdot Ub \cdot m_X^{\mathrm{op}}$ between ultrarelations (i.e. relations between ultrafilters and points) which has the inverse image relation $e_X^{\mathrm{op}} : UX \nrightarrow X$ of the function $e_X : X \to UX$ as a (lax) identity. Using this composition we can present topologies as monoids as well: an ultrarelation $a : UX \nrightarrow X$ (which can be considered as an endomorphism of $X$ in the co-Kleisli (lax) category) is the convergence structure of a topology precisely if

$$e_X^{\mathrm{op}} \leqslant a, \quad a * a \leqslant a.$$

Replacing **Rel** by another suitable 2-category as well as $\mathsf{U}$ by a suitable monad $\mathsf{T}$ which has a lax extension to this 2-category, we obtain further interesting categories as categories of lax Eilenberg–Moore algebras such as (generalized) metric spaces and approach spaces. In order to capture all these examples, [7] develops the notion of $(\mathsf{T};\mathbf{V})$-algebras for a complete, cocomplete, symmetric monoidal closed category $\mathbf{V}$ and a $\mathbf{V}$-admissible monad $\mathsf{T} = (T, e, m)$. The general framework of [7], with $\mathbf{V}$ being a lattice, will be our basic setting.

In this setting, a characterization of regular epimorphisms can be obtained by a standard argument. We forget first the transitivity axiom and hence work in the larger category of reflexive lax algebras, of which transitive structures form a reflective full subcategory (see [3], for instance). There, regular epimorphisms are exactly those lax homomorphisms which are surjective on both points and structure. A lax homomorphism $f : (X, a) \to (Y, b)$ between transitive structures is a regular epimorphism if and only if it is surjective and the structure $b$ on $Y$ is the transitive reflection of the (not-necessarily transitive) image structure of $f$. Now the standard description of this reflection—as the largest element of the chain $b_\alpha$ of structures on $Y$ where $b_{\alpha+1} = b_\alpha * b_\alpha$—does not give an elegant result since in each step we use both $b$ and $m_Y$, and hence the reflection is a mixture of $b$-terms and $m$-terms. In this paper, we present an improvement of this description where these terms are separated. This improvement gives indeed the expected characterization of regular epimorphisms.

## 1. $(\mathsf{T};\mathbf{V})$-algebras

### *1.1.* $\mathbf{V}$*-matrices*

Throughout $\mathbf{V}$ denotes a *symmetric monoidal closed complete lattice,* with tensor $\otimes$ and neutral element $k$. Important examples are the two-element chain $\mathbf{2} = \{\mathsf{false} \vdash \mathsf{true}\}$ with tensor given by "and" & and neutral element $\mathtt{true}$; and the extended real half-line $\overline{\mathbb{R}}_+ = [0, \infty]$ ordered by the "greater or equal"-relation $\geqslant$, with tensor given by addition (where $\infty + x = x + \infty = \infty$) and 0 as neutral element.

The category $\mathrm{Mat}(\mathbf{V})$ of $\mathbf{V}$-*matrices* has sets as its objects, and a morphism $r : X \nrightarrow Y$ in $\mathrm{Mat}(\mathbf{V})$ is a $\mathbf{V}$-matrix $r : X \times Y \to \mathbf{V}$. Composition of $\mathbf{V}$-matrices $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ is defined as matrix multiplication

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

and the $\mathbf{V}$-matrix $\mathrm{id}_X : X \nrightarrow X$, which sends all diagonal elements $(x, x)$ to $k$ and all other elements to the bottom element $\perp$ of $\mathbf{V}$, acts as an identity. The order of $\mathbf{V}$ induces a complete order relation on $\mathrm{Mat}(\mathbf{V})(X, Y)$: for $\mathbf{V}$-matrices $r, r' : X \nrightarrow Y$ we define

$$r \leqslant r' \;:\iff\; \forall x \in X \; \forall y \in Y \quad r(x, y) \leqslant r'(x, y).$$

This order relation is preserved by composition. Therefore $\mathrm{Mat}(\mathbf{V})$ is actually a 2-*category*. In addition, composition preserves suprema in each variable since $\otimes$ does, that is

$$\bigvee_{j\in J, i\in I} s_j \cdot r_i = \bigvee_{j\in J} s_j \cdot \bigvee_{i\in I} r_i.$$

$\mathrm{Mat}(\mathbf{V})$ has an order-preserving *involution* $\_^{\mathrm{op}}$ sending each $r : X \nrightarrow Y$ to its transpose $r^{\mathrm{op}} : Y \nrightarrow X$ defined by $r^{\mathrm{op}}(y, x) = r(x, y)$. This involution induces a contravariant 2-endofunctor on $\mathrm{Mat}(\mathbf{V})$.

### 1.2. *Examples*

$\mathrm{Mat}(\mathbf{2}) \cong \mathbf{Rel}$ and a morphism $a : X \nrightarrow Y$ of $\mathrm{Mat}(\overline{\mathbb{R}}_+)$ is a generalized distance $a : X \times Y \to \overline{\mathbb{R}}_+$. Composition in $\mathrm{Mat}(\overline{\mathbb{R}}_+)$ is given by

$$b \cdot a(x, z) = \inf\{a(x, y) + b(y, z) \mid y \in Y\},$$

$\mathrm{id}_X : X \nrightarrow X$ is the discrete distance sending the diagonal to 0 and all other pairs $(x, x')$ to $\infty$.

### 1.3. *Embedding* **Set**

There is a natural embedding of **Set** into $\mathrm{Mat}(\mathbf{V})$ leaving objects unchanged and sending each map $f : X \to Y$ to the $\mathbf{V}$-matrix

$$f(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{else.} \end{cases}$$

In the sequel we will write $f : X \to Y$ rather then $f : X \nrightarrow Y$ for a $\mathbf{V}$-matrix induced by a **Set**-map in the sense above. We remark that each $f : X \to Y$ satisfies the inequations $\mathrm{id}_X \leqslant f^{\mathrm{op}} \cdot f$ and $f \cdot f^{\mathrm{op}} \leqslant \mathrm{id}_Y$, i.e. $f$ is left adjoint to $f^{\mathrm{op}}$.

### 1.4. $\mathbf{V}$-*admissible monads*

A *monad* $\mathsf{T} = (T, e, m)$ on **Set** is called $\mathbf{V}$-*admissible* if the endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ admits an extension to $\mathrm{Mat}(\mathbf{V})$ such that

(i) $Ts \cdot Tr \leqslant T(s \cdot r)$,
(ii) $r \leqslant r' \;\Rightarrow\; Tr \leqslant Tr'$,
(iii) $e_Y \cdot r \leqslant Tr \cdot e_X$,
(iv) $m_Y \cdot T^2 r \leqslant Tr \cdot m_X$,
(v) $(Tr)^{\mathrm{op}} = T(r^{\mathrm{op}})$ (and we write $Tr^{\mathrm{op}}$)

for all $r, r' : X \nrightarrow Y$ and $s : Y \nrightarrow Z$. We remark that (i) becomes an equality in case $r = f$ is a map, i.e. $T$ preserves composition of $\mathbf{V}$-matrices with maps from the right. A $\mathbf{V}$-admissible monad may have more than one extension (see [6]). *From now on we fix an extension when considering a* $\mathbf{V}$-*admissible monad* $\mathsf{T}$.

### 1.5. Examples

The identity monad $1 = (\mathrm{Id}, \mathrm{id}, \mathrm{id})$ on **Set** can be obviously "extended" to the identity monad on $\mathrm{Mat}(\mathbf{V})$ and hence is **V**-admissible. In the sequel we will only consider this canonical extension of 1.

The ultrafilter monad $\mathsf{U} = (U, e, m)$ on **Set** is induced by the dual adjunction

$$\mathbf{Bool} \overset{\eta}{\Rightarrow} \underset{\hom(\_,2)}{\overset{\hom(\_,2)}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathbf{Set}.$$

Explicitly, the ultrafilter functor $U : \mathbf{Set} \to \mathbf{Set}$ sends each set $X$ to the set $UX$ of its ultrafilters and each function $f : X \to Y$ to the function $Uf : UX \to UY$, which takes an ultrafilter $\mathfrak{x} \in UX$ to the (ultra)filter generated by its $f$-image $\{f[A] \mid A \in \mathfrak{x}\}$. The natural transformations $e$ and $m$ are given by

$$e_X(x) = \dot{x} = \{A \subset X \mid x \in A\} \quad \text{and} \quad m_X(\mathfrak{X}) = \{A \subset X \mid A^{\#} \in \mathfrak{X}\}$$

for all $\mathfrak{X} \in U^2X$ and $x \in X$. Here $A^{\#}$ denotes the set $\{\mathfrak{a} \in UX \mid A \in \mathfrak{a}\}$. In the sequel we will extend this notation to a filter $\mathfrak{f}$ on $X$ and write $\mathfrak{f}^{\#}$ for the filter base $\{A^{\#} \mid A \in \mathfrak{f}\}$. It is shown in [1] that the ultrafilter monad $\mathsf{U}$ is in a canonical way **2**-admissible and in [7] this result is extended to a more general class of lattices **V** including $\mathbf{V} = \overline{\mathbb{R}}_+$. We remark that $m$ becomes a (strict) natural transformation for these extensions and that $U$ extends to a (strict) functor to $\mathbf{Rel} \cong \mathrm{Mat}(\mathbf{2})$.

### 1.6. $(\mathsf{T}; \mathbf{V})$-algebras

Given now a **V**-admissible monad $\mathsf{T} = (T, e, m)$, the category $\mathrm{Alg}(\mathsf{T}; \mathbf{V})$ of $(\mathsf{T}; \mathbf{V})$-*algebras* has as its objects pairs $(X, a)$ consisting of a set $X$ and a structure $a : TX \nrightarrow X$ in $\mathrm{Mat}(\mathbf{V})$ satisfying the reflexivity and transitivity laws

$$\text{(Refl) } \mathrm{id}_X \leqslant a \cdot e_X \quad \text{(Trans) } a \cdot Ta \leqslant a \cdot m_X.$$

A morphism $f : (X, a) \to (Y, b)$ in $\mathrm{Alg}(\mathsf{T}; \mathbf{V})$ is a *lax homomorphism*, that is, a map $f : X \to Y$ such that $f \cdot a \leqslant b \cdot Tf$.

### 1.7. Examples

$(\mathsf{T} = 1, \mathbf{V} = \mathbf{2})$: A $(1; \mathbf{2})$-algebra is a pair $(X, R)$ consisting of a set $X$ and a binary relation $R$ on $X$, the two basic axioms read as

$$\mathsf{true} \vdash xRx \quad (xRy\ \&\ yRz) \vdash xRz.$$

Moreover, a lax homomorphism is an order-preserving map. Hence $\mathrm{Alg}(1; \mathbf{2})$ is isomorphic to the category **Ord** of preordered sets.

($\mathsf{T} = 1$, $\mathbf{V} = \overline{\mathbb{R}}_+$): A $(1; \overline{\mathbb{R}}_+)$-algebra is a set $X$ together with a (generalized) distance $d : X \times X \to \overline{\mathbb{R}}_+$ satisfying

$$0 \geqslant d(x,x), \quad d(x,y) + d(y,z) \geqslant d(x,z).$$

A lax homomorphism is a non-expanding map. We denote the resulting category by **Met**.

($\mathsf{T} = 1$): More general, $(1; \mathbf{V})$-algebras are exactly the categories enriched over $\mathbf{V}$ and lax homomorphisms are $\mathbf{V}$-functors (see [11]).

($\mathsf{T} = \mathsf{U}$, $\mathbf{V} = \mathbf{2}$): The main result of [1] states that $\mathrm{Alg}(\mathsf{U}; \mathbf{2}) \cong \mathbf{Top}$.

($\mathsf{T} = \mathsf{U}$, $\mathbf{V} = \overline{\mathbb{R}}_+$): It is shown in [3] that $(\mathsf{U}; \overline{\mathbb{R}}_+)$-algebras coincide with approach spaces in the sense of Lowen [12] and lax homomorphisms with non-expanding maps.

### 1.8. *Reflexive algebras*

Many constructions such as forming function spaces cannot be done within topological spaces, being often useful to move temporarily into the cartesian closed category of pseudotopological spaces (see [8]). Here a pseudotopology on a set $X$ is a relation $a : UX \nrightarrow X$, which is only required to fulfil the reflexivity law $\dot{x} \to x$. In the setting of $(\mathsf{T}; \mathbf{V})$-algebras a similar technique can be used: we define the category $\mathrm{Alg}(T, e; \mathbf{V})$ of *reflexive lax algebras* having as objects such pairs $(X, a)$, where $a$ is only required to fulfil the reflexivity law (Refl), and lax homomorphisms as morphisms. In [6] it is proven that—under mild assumptions—$\mathrm{Alg}(T, e; \mathbf{V})$ is locally cartesian closed. Moreover, we have that (see [3]):

**Proposition 1.** *A morphism $f : (X, a) \to (Y, b)$ in $\mathrm{Alg}(T, e; \mathbf{V})$ is a regular epimorphism if and only if $b = f \cdot a \cdot Tf^{\mathrm{op}}$.*

$\mathrm{Alg}(T, e; \mathbf{V})$ contains $\mathrm{Alg}(\mathsf{T}; \mathbf{V})$ as a full and reflective subcategory where the reflection morphism is identity carried [3]. We shall describe this reflection in Section 3. In analogy to the transitive reflection of a reflexive relation, the reflection of a reflexive structure $a : TX \nrightarrow X$ can be obtained as an “iterated composite” of $a$; here composition must be read as co-Kleisli composition.

## 2. The co-Kleisli composition

### 2.1. *Definition*

For a fixed $\mathbf{V}$-admissible monad $\mathsf{T} = (T, e, m)$, the category $\mathrm{Mat}(\mathbf{V})$ has an important additional structure: the *co-Kleisli composition* defined as

$$a * b = a \cdot Tb \cdot m_Z^{\mathrm{op}}$$

for all $b : TZ \nrightarrow Y$ and $a : TY \nrightarrow X$. As it is already observed in [5], this is indeed the Kleisli composition for the lax comonad $(T, e^{\mathrm{op}}, m^{\mathrm{op}})$ on $\mathrm{Mat}(\mathbf{V})$. It follows from the definition that $*$ preserves suprema on the left-hand side since the ordinary composition of $\mathrm{Mat}(\mathbf{V})$

does so:

$$\bigvee_{i\in I} a_i * a = \left(\bigvee_{i\in I} a_i\right) * a.$$

The **V**-matrix $e_X^{\mathrm{op}}$ acts as a (lax) identity: $a * e_Y^{\mathrm{op}} = a$ and $e_X^{\mathrm{op}} * a \geqslant a$, where in the latter inequation becomes an equality whenever $e$ extends to a (strict) natural transformation. Moreover, we have

$$a * (b * c) \leqslant (a * b) * c$$

provided that $T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ is a functor, and

$$a * (b * c) \geqslant (a * b) * c$$

whenever $m$ extends to a (strict) natural transformation.

### 2.2. $(\mathsf{T}; \mathbf{V})$-*algebras as monoids*

Using the co-Kleisli composition, we can express the two fundamental laws—reflexivity and transitivity—of an $(\mathsf{T}; \mathbf{V})$-algebra $(X, a)$ as a monoid structure on $a$: they are equivalent to

$$e_X^{\mathrm{op}} \leqslant a, \quad a * a \leqslant a.$$ [1]

This description will be the key to our study of the transitive reflection of a reflexive structure in the next section. Before we do so, we shall have a closer look at a special example.

### 2.3. *Co-Kleisli composition for the ultrafilter monad*

As already mentioned in (1.5), the ultrafilter functor $U : \mathbf{Set} \to \mathbf{Set}$ can be extended to an endofunctor on **Rel** such that $e : \mathrm{Id} \to U$ extends to a op-lax natural transformation and $m : U^2 \to U$ to a (strict) natural transformation. Explicitly, for a relation $r : X \nrightarrow Y$ we define $Ur : UX \nrightarrow UY$ by $\mathfrak{x}(Ur)\mathfrak{y} :\iff r[\mathfrak{x}] \subset \mathfrak{y} \iff r^{\mathrm{op}}[\mathfrak{y}] \subset \mathfrak{x}$. We shall make use of the *Zariski closure* on *UX* which is defined by $\mathfrak{x} \in \mathrm{cl}\,\mathscr{A} :\iff \mathfrak{x} \supset \bigcap \mathscr{A}$ for $\mathfrak{x} \in UX$ and $\mathscr{A} \subset UX$, which can be equivalently expressed by $\mathfrak{x} \subset \bigcup \mathscr{A}$. Our next result characterizes those relations $a : UY \nrightarrow X$ for which $e_X^{\mathrm{op}}$ acts as an identity.

**Proposition 2.** *Let* $a : UY \nrightarrow X$. *The following assertions are equivalent.*

(1) $e_X^{\mathrm{op}} * a = a$.
(2) *For each* $x \in X$, $a^{\mathrm{op}}(x) = \{\mathfrak{y} \in UY \mid \mathfrak{y} a x\}$ *is closed in UY with respect to the Zariski closure.*

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[1] Note that $e_X^{\mathrm{op}} \leqslant$ a implies already $a \leqslant a * a$.

**Proof.** It follows easily from the fact that, for each $\mathfrak{y} \in UY$ and each $x \in X$, it holds

$$\mathfrak{y} \subset \bigcup a^{\mathrm{op}}(x) \iff \exists \mathfrak{Y} \in U^2Y \quad (m_Y(\mathfrak{Y}) = \mathfrak{y} \ \&\ \mathfrak{Y}(Ua)\dot{x}). \qquad \square$$

Given now sets $X$ and $Y$, each relation $a : UY \nrightarrow X$ defines a function

$$\psi(a) : PY \to PX, \quad M \mapsto a[M^{\#}] = \{x \in X \mid \exists \mathfrak{y} \in UY \quad (\mathfrak{y} a x \ \&\ M \in \mathfrak{y})\}$$

and, conversely, each function $c : PY \to PX$ defines a relation $\phi(c) : UY \nrightarrow X$ by

$$\mathfrak{y}\phi(c)x \ :\iff \ \forall M \in \mathfrak{y} \ \ x \in c(M).$$

We obtain a pair of order-preserving functions

$$\{\text{functions } c : PY \to PX\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \{\text{relations } a : UY \nrightarrow X\}.^2$$

It is easy to see that $\psi\phi \leqslant \mathrm{id}$ and $\mathrm{id} \leqslant \phi\psi$, hence $\psi$ is left adjoint to $\phi$. The following proposition identifies the fixed objects of this adjunction.

**Proposition 3.** (1) *For each $a : UY \nrightarrow X$, $\phi\psi(a) = e_X^{\mathrm{op}} * a$.*

(2) *For each $c : PY \to PX$, $c = \psi\phi(c)$ if and only if $c$ is additive, i.e. preserves finite unions.*

**Proof.** First, let $\mathfrak{y} \in UY$ and $x \in X$. It holds

$$\begin{aligned} \mathfrak{y}(e_X^{\mathrm{op}} * a)x &\iff \exists \mathfrak{Y} \in U^2Y \quad (\mathfrak{Y}(Ua)\dot{x} \ \&\ m_Y(\mathfrak{Y}) = \mathfrak{y}) \\ &\iff \forall M \in \mathfrak{y} \ \ x \in a[M^{\#}] \\ &\iff \forall M \in \mathfrak{y} \ \ x \in \psi(a)(M) \\ &\iff \mathfrak{y}\phi\psi(a)x. \end{aligned}$$

It is easy to see that each function of the form $\psi(a)$ is additive. Assume now that $c : PY \to PX$ is additive. We have to show that $c \leqslant \psi\phi(c)$. To do so, let $M \subset Y$ and $x \in X$ be such that $x \in c(M)$. Since $c$ is additive,

$$\mathfrak{i} = \{N \subset Y \mid x \notin c(N)\}$$

is an ideal which does not contain $M$. Therefore there exists an ultrafilter $\mathfrak{y} \in UY$ containing $M$ and disjoint from $\mathfrak{i}$. Hence $\mathfrak{y}\phi(c)x$ and consequently $x \in \psi\phi(c)(M)$. $\square$

**Proposition 4.** *For all $d : PZ \to PY$, $c : PY \to PX$ and $b : UZ \nrightarrow Y$, $a : UY \nrightarrow X$,*

$$\begin{aligned} &\psi(e_X^{\mathrm{op}}) = \mathrm{id}_{PX}, \quad \psi(a * b) = \psi(a) \cdot \psi(b), \\ &\phi(\mathrm{id}_{PX}) = e_X^{\mathrm{op}}, \quad \phi(c \cdot d) \geqslant \phi(c) * \phi(d); \end{aligned}$$

*with equality whenever $c$ and $d$ are additive.*

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[2] More precisely, we should write $\phi_{Y,X}$ and $\psi_{Y,X}$. For the sake of simplicity we omit the indexes.

**Proof.** The equalities $\phi(\mathrm{id}_{PX}) = e_X^{\mathrm{op}}$ and $\psi(e_X^{\mathrm{op}}) = \mathrm{id}_{PX}$ hold obviously.

Assume first that $x \in \psi(a * b)(M)$. Hence there exist $\mathfrak{Z} \in U^2Z$ and $\mathfrak{y} \in UY$ such that

$$M^\# \in \mathfrak{Z}, \quad \mathfrak{Z}(Ub)\mathfrak{y} \quad \text{and} \quad \mathfrak{y}ax.$$

Therefore $\psi(b)(M) = b[M^\#] \in \mathfrak{y}$ and consequently $x \in \psi(a)(\psi(b)(M))$.

Assume now that $x \in \psi(a)(\psi(b)(M))$. Hence there exists $\mathfrak{y} \in UY$ with $\mathfrak{y}ax$ and $b[M^\#] = \psi(b)(M) \in \mathfrak{y}$. Therefore we can find $\mathfrak{Z} \in UZ$ with $M^\# \in \mathfrak{Z}$ and $\mathfrak{Z}(Ub)\mathfrak{y}$, which implies $x \in \psi(a * b)(M)$.

Assume now that $\mathfrak{z}(\phi(c) * \phi(d))x$, that is, there exist $\mathfrak{Z} \in U^2Z$ and $\mathfrak{y} \in UY$ such that

$$\mathfrak{z} = m_X(\mathfrak{Z}), \quad \mathfrak{Z}U\phi(d)\mathfrak{y} \quad \text{and} \quad \mathfrak{y}\phi(c)x.$$

Hence $d(M) \supset \psi\phi(d)(M) = \phi(d)[M^\#] \in \mathfrak{y}$ for each $M \in \mathfrak{z}$, which, together with $\mathfrak{y}\phi(c)x$, implies

$$\forall M \in \mathfrak{z} \;\; x \in c(d(M)).$$

Finally, assume that $c$ and $d$ are additive functions and that $\mathfrak{z}\phi(c \cdot d)x$. Hence for all $M \in \mathfrak{z}$ we have $x \in c(d(M))$. This together with additivity of $c$ and $d$ implies that

$$\mathfrak{f} = \{d(M) \mid M \in \mathfrak{z}\}$$

is a filterbase disjoint from the ideal

$$\mathfrak{i} = \{N \subset Y \mid x \notin c(N)\}.$$

Therefore there exists $\mathfrak{y} \in UY$ containing $\mathfrak{f}$ and disjoint from $\mathfrak{i}$, hence $\mathfrak{y}\phi(c)x$. From $d(M) = \psi\phi(d)(M) = \phi(d)[M^\#]$ we deduce $\phi(d)[\mathfrak{z}^\#] = \mathfrak{f}$. Consequently there exists $\mathfrak{Z} \in U^2Z$ containing $\mathfrak{z}^\#$ such that $\mathfrak{Z}U\phi(d)\mathfrak{y}$. We conclude then that $\mathfrak{z}(\phi(c) * \phi(d))x$. $\square$

## 3. The transitive reflection

### 3.1. Description of the reflection

As it is already worked out in [1] (for $\mathbf{V} = \mathbf{2}$) and [3], the transitive reflection of a reflexive lax algebra $(X, a)$ for a given $\mathbf{V}$-admissible monad $(T, e, m)$ can be obtained by the following transfinite process: we define an ascending chain of $\mathrm{Mat}(\mathbf{V})$-morphisms $\hat{a}_\alpha : TX \nrightarrow X$ ($\alpha$ any ordinal larger then 0) by putting

$$\hat{a}_1 = a, \quad \hat{a}_{\alpha+1} = \hat{a}_\alpha * \hat{a}_\alpha, \quad \hat{a}_\lambda = \bigvee_{\alpha<\lambda} \hat{a}_\alpha.$$

Since there is only a set of functions from $TX \times X$ to $\mathbf{V}$, there must exist an ordinal $\gamma$ such that $\hat{a}_{\gamma+1} = \hat{a}_\gamma$. This $\hat{a}_\gamma$ is obviously transitive and $(X, \hat{a}_\gamma)$ is indeed the transitive reflection of $(X, a)$.

Besides the exponential growing of the number of terms in this iteration process, it has another disadvantage for our purpose: it gives us a structure $\hat{a}_\gamma$ which is a mixture of $a$- and *m*-terms. We will now describe an alternative iteration process where in the induction step *a*-terms are only inserted on the left and *m*-terms on the right-hand side. Concretely, we will consider $a_{\alpha+1} = a * a_\alpha$ instead of $\hat{a}_{\alpha+1} = \hat{a}_\alpha * \hat{a}_\alpha$ and then show that $a_\alpha = a^\alpha \cdot (\mu_X^\alpha)^{\mathrm{op}}$ where $a^\alpha$ is obtained as an iteration of $a$ and $\mu_X^\alpha$ as an iteration of $m_X$. To do so, we shall use lax associativity of the co-Kleisli composition and therefore assume from now on that *m extends to a* (*strict*) *natural transformation*.

Let $(X, a)$ be a reflexive lax algebra. We define an ascending chain of **V**-matrices $a_\alpha : TX \nrightarrow X$ ($\alpha$ any ordinal) by putting

$$a_0 = e_X^{\mathrm{op}}, \quad a_{\alpha+1} = a * a_\alpha, \quad a_\lambda = \bigvee_{\alpha<\lambda} a_\alpha.$$

As before, there must exist an ordinal $\gamma$ such that $a_{\gamma+1} = a_\gamma$.

**Lemma 5.** *For all ordinals* $\alpha, \beta > 0$: $a_\beta * a_\alpha \leqslant a_{\alpha+\beta}$.

**Proof.** Let $\alpha$ be any ordinal larger than 0. For $\beta = 1$ we have

$$a_{\alpha+1} = a * a_\alpha = a_1 * a_\alpha.$$

Assume now $a_\beta * a_\alpha \leqslant a_{\alpha+\beta}$ for an ordinal $\beta > 0$. It implies

$$a_{\alpha+(\beta+1)} = a_{(\alpha+\beta)+1} = a * a_{\alpha+\beta} \geqslant a * (a_\beta * a_\alpha) \geqslant (a * a_\beta) * a_\alpha = a_{\beta+1} * a_\alpha.$$

Finally, let $\lambda$ be a limit ordinal such that the assertion is true for all $\beta < \lambda$. We obtain

$$a_{\alpha+\lambda} = \bigvee_{\beta<\lambda} a_{\alpha+\beta} \geqslant \bigvee_{\beta<\lambda} a_\beta * a_\alpha = \left(\bigvee_{\beta<\lambda} a_\beta\right) * a_\alpha = a_\lambda * a_\alpha. \qquad \square$$

Hence we have $a_\alpha * a_\alpha \leqslant a_{\alpha+\alpha}$ for each ordinal $\alpha$. As a consequence we obtain that $a_\gamma$ is transitive: $a_\gamma * a_\gamma \leqslant a_{\gamma+\gamma} = a_\gamma$. It is easy to see that $\mathrm{id}_X : (X, a) \to (X, a_\gamma)$ has indeed the required universal property and therefore it is the transitive reflection of $(X, a)$.

### *3.2. Separation of terms*

Our final goal in this section is to separate the *a*-part and the *m*-part in $a_\gamma$. More precisely, we give a presentation $a_\gamma = a^\gamma \cdot (\mu_X^\gamma)^{\mathrm{op}}$ with a **V**-matrix $a^\gamma : T^\gamma X \nrightarrow X$ coming from an iteration of $a$ and a natural transformation $\mu^\gamma : T^\gamma \to T$ obtained from an iteration of $m$. To do so, we assume from now on that $T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ *preserves composition of* **V**-*matrices with maps from the left*. We define, for all ordinals $\alpha \leqslant \beta$, functors $T^\alpha : \mathbf{Set} \to \mathbf{Set}$ and natural transformations $e^{\alpha,\beta} : T^\alpha \to T^\beta$ by putting

$$T^0 = \mathrm{Id},$$
$$T^{\alpha+1} = TT^\alpha, \quad e^{\alpha,\alpha+1} = e_{T^\alpha},$$
$$T^\lambda = \operatorname*{colim}_{\alpha<\lambda} T^\alpha, \quad e^{\alpha,\lambda} = \text{colimit injection}.$$

Moreover, for each ordinal $\alpha$ we define a natural transformation $\mu^\alpha : T^\alpha \to T$ by putting

$$\mu^0 = e, \quad \mu^{\alpha+1} = m \cdot T\mu^\alpha, \quad \mu^\lambda = [\mu^\alpha]_{\alpha<\lambda}.$$

Note that in the limit step we make use of the fact that $(\mu^\alpha)_{\alpha<\lambda}$ forms a compatible cone, i.e.

$$\mu^{\alpha+1} \cdot e^{\alpha,\alpha+1} = m \cdot T\mu^\alpha \cdot e_{T^\alpha} = m \cdot e_T \cdot \mu^\alpha = \mu^\alpha.$$

$T^\lambda X = \mathrm{colim}_{\alpha<\lambda} T^\alpha X$ is also a *lax colimit* in $\mathrm{Mat}(\mathbf{V})$ in the following sense. For any family $(c_\alpha : T^\alpha X \nrightarrow Z)_{\alpha<\lambda}$ satisfying $c_{\alpha+1} \cdot e_X^{\alpha,\alpha+1} \geqslant c_\alpha$, there is a $\mathbf{V}$-matrix $c : T^\lambda X \nrightarrow Z$ such that $c \geqslant c_\alpha$ for each ordinal $\alpha < \lambda$. Moreover, $c$ is universal with this property: it holds $c \leqslant c'$ for any $c' : T^\lambda X \nrightarrow Z$ such that $c' \geqslant c_\alpha$ for each ordinal $\alpha < \lambda$. Explicitly, $c$ is given by

$$c(\mathfrak{X}, x) = \bigvee_{\alpha<\lambda} \bigvee_{\substack{\mathfrak{a}\in T^\alpha X,\\ e^{\alpha,\lambda}(\mathfrak{a})=\mathfrak{X}}} a^\alpha(\mathfrak{a}, x)$$

for each $\mathfrak{X} \in T^\lambda X$ and $x \in X$.

**Lemma 6.** *Let $\lambda$ be a limit ordinal and assume that the following data is given.*

(1) *A diagram $(X_\alpha \xrightarrow{e^{\alpha,\beta}} X_\beta)_{\alpha\leqslant\beta<\lambda}$ in* **Set** *with colimit cone $(X_\alpha \xrightarrow{e^{\alpha,\lambda}} X_\lambda)_{\alpha<\lambda}$.*
(2) *A compatible cone $(X_\alpha \xrightarrow{h_\alpha} Y)_{\alpha<\lambda}$ with induced map $h_\lambda : X_\lambda \to Y$.*
(3) *A lax compatible cone $(X_\alpha \overset{r_\alpha}{\nrightarrow} Z)_{\alpha<\lambda}$ with induced* $\mathbf{V}$-matrix $r_\lambda : X_\lambda \nrightarrow Z$.
*Then $r_\lambda \cdot h_\lambda^{\mathrm{op}} = \bigvee_{\alpha<\lambda} r_\alpha \cdot h_\alpha^{\mathrm{op}}$.*

**Proof.** Let $y \in Y$ and $z \in Z$. We have

$$\begin{aligned} r_\lambda \cdot h_\lambda^{\mathrm{op}}(y, z) &= \bigvee_{\substack{x\in X_\lambda,\\ h_\lambda(x)=y}} r_\lambda(x, z) \\ &= \bigvee_{\substack{x\in X_\lambda,\\ h_\lambda(x)=y}} \bigvee_{\alpha<\lambda} \bigvee_{\substack{a\in X_\alpha,\\ e^{\alpha,\lambda}(a)=x}} r_\alpha(a, z) \\ &= \bigvee_{\alpha<\lambda} \bigvee_{\substack{a\in X_\alpha,\\ h_\alpha(a)=y}} r_\alpha(a, z) \quad (\text{since } h_\lambda \cdot e^{\alpha,\lambda} = h_\alpha) \\ &= \bigvee_{\alpha<\lambda} r_\alpha \cdot h_\alpha^{\mathrm{op}}(y, z). \qquad \square \end{aligned}$$

Let $(X, a)$ be a reflexive lax algebra. For each ordinal $\alpha$ we define a $\mathbf{V}$-matrix $a^\alpha : T^\alpha X \nrightarrow X$ by putting

$$a^0 = \mathrm{id}_X, \quad a^{\alpha+1} = a \cdot Ta^\alpha, \quad a^\lambda = [a^\alpha]_{\alpha<\lambda}.$$

In the limit step we make use of the fact that $(a^\alpha)_{\alpha<\lambda}$ forms a lax natural transformation, i.e.

$$a^{\alpha+1} \cdot e_X^{\alpha,\alpha+1} = a \cdot Ta^\alpha \cdot e_{T^\alpha} \geqslant a \cdot e_X \cdot a^\alpha \geqslant a^\alpha.$$

**Proposition 7.** *For each ordinal* $\alpha$, $a_\alpha = a^\alpha \cdot (\mu_X^\alpha)^{\mathrm{op}}$.

**Proof.** It holds $a_0 = e_X^{\mathrm{op}} = \mathrm{id}_X \cdot (\mu_X^0)^{\mathrm{op}}$. Assume now that the assertion is true for an ordinal $\alpha$. Then we have

$$\begin{aligned} a_{\alpha+1} &= a * a_\alpha \\ &= a * (a^\alpha \cdot (\mu_X^\alpha)^{\mathrm{op}}) \\ &= a \cdot Ta^\alpha \cdot T(\mu_X^\alpha)^{\mathrm{op}} \cdot m_X^{\mathrm{op}} \\ &= a^{\alpha+1} \cdot (\mu_X^{\alpha+1})^{\mathrm{op}}. \end{aligned}$$

Finally, let $\lambda$ be a limit ordinal and assume that the assertion is true for each ordinal $\alpha < \lambda$. Applying Lemma 6 we obtain

$$a_\lambda = \bigvee_{\alpha<\lambda} a_\alpha = \bigvee_{\alpha<\lambda} a^\alpha \cdot (\mu_X^\alpha)^{\mathrm{op}} = a^\lambda \cdot (\mu_X^\lambda)^{\mathrm{op}}. \qquad \square$$

## 4. The characterization of regular epimorphisms

### 4.1. "Zigzags"

A regular epimorphism in $\mathrm{Alg}(\mathsf{T};\mathbf{V})$ may fail the condition of Proposition 1 simply because $f \cdot a \cdot Tf^{\mathrm{op}}$ need not be transitive. However, we have

**Proposition 8.** *A morphism* $f : (X, a) \to (Y, b)$ *in* $\mathrm{Alg}(\mathsf{T};\mathbf{V})$ *is a regular epimorphism if and only if f is surjective and b is the transitive reflection of the reflexive structure* $f \cdot a \cdot Tf^{\mathrm{op}}$.

According to the previous section, this reflection is given by $(f \cdot a \cdot Tf^{\mathrm{op}})_\gamma = (f \cdot a \cdot Tf^{\mathrm{op}})^\gamma \cdot (\mu_Y^\gamma)^{\mathrm{op}}$ for some ordinal $\gamma$. Our final aim is to present the first component as the image of a "zigzag" on $X$.

Let $(X, a)$ be a reflexive lax algebra and $f : X \to Y$ a map. For each ordinal $\alpha$ we define a "zigzag" structure $a_f^\alpha : T^\alpha X \nrightarrow X$ by putting

$$a_f^0 = \mathrm{id}_X, \quad a_f^{\alpha+1} = a \cdot \underbrace{Tf^{\mathrm{op}} \cdot Tf}_{=\ker(Tf)} \cdot Ta_f^\alpha, \quad a_f^\lambda = [a_f^\alpha]_{\alpha<\lambda}.$$

As before, in the limit step we make use of the fact that $(a_f^\alpha)_{\alpha<\lambda}$ forms a lax natural transformation, i.e.

$$a_f^{\alpha+1} \cdot e_X^{\alpha,\alpha+1} = a \cdot Tf^{\mathrm{op}} \cdot Tf \cdot Ta_f^\alpha \cdot e_{T^\alpha X} \geqslant a \cdot \mathrm{id}_{TX} \cdot e_X \cdot a_f^\alpha \geqslant a_f^\alpha.$$

**Proposition 9.** *For each ordinal $\alpha$ and surjective f, it holds*

$$(f \cdot a \cdot Tf^{\mathrm{op}})^{\alpha} = f \cdot a_f^{\alpha} \cdot T^{\alpha} f^{\mathrm{op}}.$$

**Proof.** For $\alpha = 0$ we have

$$f \cdot a_f^0 \cdot T^0 f^{\mathrm{op}} = f \cdot \mathrm{id}_X \cdot f^{\mathrm{op}} = \mathrm{id}_Y = (f \cdot a \cdot Tf^{\mathrm{op}})^0.$$

Assume now that the assertion is true for an ordinal $\alpha$. Then it holds

$$\begin{aligned}(f \cdot a \cdot Tf^{\mathrm{op}})^{\alpha+1} &= (f \cdot a \cdot Tf^{\mathrm{op}}) \cdot T((f \cdot a \cdot Tf^{\mathrm{op}})^{\alpha})\\ &= f \cdot a \cdot Tf^{\mathrm{op}} \cdot Tf \cdot Ta_f^{\alpha} \cdot T^{\alpha+1} f^{\mathrm{op}}\\ &= f \cdot a_f^{\alpha+1} \cdot T^{\alpha+1} f^{\mathrm{op}}.\end{aligned}$$

Finally, let $\lambda$ be a limit ordinal and assume that the assertion is true for each ordinal $\alpha < \lambda$. An application of Lemma 6 gives

$$\begin{aligned}(f \cdot a \cdot Tf^{\mathrm{op}})^{\lambda} &= [(f \cdot a \cdot Tf^{\mathrm{op}})^{\alpha}]_{\alpha<\lambda} = [f \cdot a_f^{\alpha} \cdot T^{\alpha} f^{\mathrm{op}}]_{\alpha<\lambda}\\ &= f \cdot [a_f^{\alpha}]_{\alpha<\lambda} \cdot T^{\lambda} f^{\mathrm{op}}. \qquad \square\end{aligned}$$

### 4.2. *The characterization*

Putting everything together we have proved

**Theorem 10.** *Let $(T, e, m)$ be a* **V***-admissible* **Set***-monad and assume that m extends to a (strict) natural transformation and $T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ (strictly) preserves composition of* **V***-matrices with maps from the left. Then $f : (X, a) \to (Y, b)$ in $\mathrm{Alg}(\mathsf{T}; \mathbf{V})$ is a regular epimorphism if and only if there exists an ordinal $\gamma$ such that*

$$b = f \cdot a_f^{\gamma} \cdot T^{\gamma} f^{\mathrm{op}} \cdot (\mu_Y^{\gamma})^{\mathrm{op}} = f \cdot a_f^{\gamma} \cdot (\mu_X^{\gamma})^{\mathrm{op}} \cdot Tf^{\mathrm{op}}.$$

## 5. Examples

### 5.1. **V**-*categories*

We consider first $\mathsf{T} = 1$. Since the co-Kleisli composition coincides with the ordinary composition, the transitive reflection of a reflexive structure $b : X \times X \to \mathbf{V}$ is given by $b^{\omega}$. Writing $x \xrightarrow{\xi} x'$ instead of $a(x, x') = \xi$, Theorem 10 implies

**Theorem 11.** *A* **V***-functor* $f:(X,a)\to(Y,b)$ *is a regular epimorphism if and only if, for each* $y_1\xrightarrow{\theta}y_0$ *in* $(Y,b)$, $\theta$ *is the supremum of all* $\xi_n\otimes\cdots\otimes\xi_1$ *obtained from "zigzags"*

$$\begin{tikzcd}
x_n \arrow[d, "\xi_n"'] \\
x_{n-1} & \sim_f & x'_{n-1} \arrow[d, "\xi_{n-1}"'] \\
& & x_{n-2} & \sim_f & x'_{n-2} \arrow[d, "\xi_{n-2}"'] \\
& & & & {} \arrow[dr, dotted, no head] \\
& & & & & {} \arrow[d, "\xi_2"'] \\
& & & & & x_1 & \sim_f & x'_1 \arrow[d, "\xi_1"'] \\
& & & & & & & x_0
\end{tikzcd}$$

*in* $(X,a)$ $(n\in\mathbb{N})$ *with* $f(x_n)=y_1$ *and* $f(x_0)=y_0$, *where* $\sim_f$ *denotes the kernel relation of f.*

Note that this applies in particular to **Met** and **Ord** (see (1.7)). In the latter case we obtain the characterization which motivated our work.

*5.2. Topological spaces*

For $\mathbf{V}=\mathbf{2}$ and $\mathsf{T}=\mathsf{U}=(U,e,m)$ the ultrafilter monad we have $\mathrm{Alg}(\mathsf{U};\mathbf{2})\cong\mathbf{Top}$. Theorem 10 specializes to

**Theorem 12.** *A continuous map* $f:X\to Y$ *in* **Top** *is a regular epimorphism if and only if there exists an ordinal* $\gamma$ *such that, for any* $\mathfrak{y}\in UY$ *and* $y\in Y$,

$$\mathfrak{y}\to y \iff \begin{cases} \textit{there exist } \mathfrak{X}\in U^{\gamma}X \textit{ and } x\in X \textit{ such that}\\ Uf\cdot\mu_X^{\gamma}(\mathfrak{X})=\mathfrak{y} \quad \& \quad f(x)=y \quad \& \quad \mathfrak{X}\,a_f^{\gamma}\,x.\end{cases}$$

Quotient maps with respect to a closure operator are characterized in [13]. We will now show how this characterization, specialized to the Kuratowski closure operator, is related to our result. Recall that a *pretopology c* on a set *X* is an additive function $c:PX\to PX$ such that $A\subset c(A)$ holds for all $A\subset X$. A *topology* is a pretopology *c* which is in addition idempotent, i.e. $c\cdot c=c$. A map $f:(X,c)\to(Y,d)$ between pretopological spaces is *continuous* if

$$f_*\cdot c\leqslant d\cdot f_*,$$

which can be equally expressed by

$$f_*\cdot c\cdot f^*\leqslant d,$$

where $f_*:PX\to PY$ is the direct image and $f^*:PY\to PX$ the inverse image function. For a pretopology *c* on *X* and a map $f:X\to Y$ we have the function $F_c=f_*\cdot c\cdot f^*:PY\to$

$PY$, that gives rise to an ascending chain of additive functions $F_c^\alpha : PY \to PY$ ($\alpha$ any ordinal) by putting

$$F_c^0 = \mathrm{id}_{PX}, \quad F_c^{\alpha+1} = F_c \cdot F_c^\alpha, \quad F_c^\lambda = \bigvee_{\alpha<\lambda} F_c^\alpha.$$

Note that $F_c$ and consequently each $F_c^\alpha$ is indeed a pretopology provided that $f$ is surjective. This iteration process must become stationary at some ordinal $\gamma$. A continuous map $f : (X, c) \to (Y, d)$ between topological spaces is a quotient map if and only if $d \leqslant F_c^\gamma$ (see [13]). In (2.3) we have shown that co-Kleisli composition of convergence structures corresponds precisely to composition of additive functions. From this the following lemma can be easily deduced.

**Lemma 13.** *Let X be a topological space, with convergence structure a and closure operator c. Let $f : X \to Y$ be a surjective map. For each ordinal $\alpha$, it holds*

(1) $F_c^\alpha = \psi((f \cdot a \cdot Uf^{\mathrm{op}})_\alpha) = f_* \cdot \psi(a_f^\alpha \cdot (\mu_X^\alpha)^{\mathrm{op}}) \cdot f^*$ *and*
(2) $\phi(F_c^\alpha) \leqslant (f \cdot a \cdot Uf^{\mathrm{op}})_{\alpha+1} = f \cdot a_f^{\alpha+1} \cdot (\mu_X^{\alpha+1})^{\mathrm{op}} \cdot Uf^{\mathrm{op}}$.

Let now $f : X \to Y$ be a continuous map. Let $a$ and $b$ denote the convergence relation of $X$ and $Y$, respectively, and $c$ and $d$ its corresponding closure operator. Then $d = F_c^\gamma$ implies

$$b = \phi(d) = \phi(F_c^\gamma) \leqslant f \cdot a_f^{\alpha+1} \cdot (\mu_X^{\alpha+1})^{\mathrm{op}} \cdot Uf^{\mathrm{op}}$$

and $b = f \cdot a_f^\gamma \cdot (\mu_X^\gamma)^{\mathrm{op}} \cdot Uf^{\mathrm{op}}$ implies

$$d = \psi(b) = \psi(f \cdot a_f^\gamma \cdot (\mu_X^\gamma)^{\mathrm{op}} \cdot Uf^{\mathrm{op}}) = f_* \cdot \psi(a_f^\gamma \cdot (\mu_X^\gamma)^{\mathrm{op}}) \cdot f^* = F_c^\gamma.$$

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